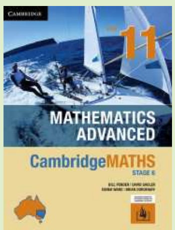
NORMANHURST BOYS HIGH SCHOOL

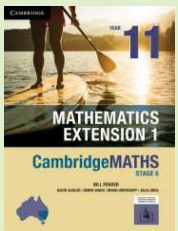
MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

With references to

Ⓐ Ⓧ1 **Discrete Probability Distributions**





Name: ....

Initial version by I. Ham, with additional suggestions from H. Lam, April 2019. Last updated October 24, 2023.
Various corrections by students & members of the Department of Mathematics at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under ⓒ CC BY 2.0.

Symbols used

Beware! Heed warning.

Mathematics Advanced content.

Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-7 uses concepts and techniques from probability to present and interpret data and solve problems in a variety of contexts, including the use of probability distributions

Syllabus subtopics

MA-S1 Discrete Probability Distributions

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 11 Advanced* (Pender, Sadler, Shea, & Ward, 2018) or *CambridgeMATHS Year 11 Extension* (Pender, Sadler, Shea, & Ward, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

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Section 1

Random Variables

This section introduces some necessary language for probability distributions. After a distinction is made between discrete and continuous probability distributions, the rest of the chapter deals only with discrete distributions, and continuous probability distributions are left until Year 12.

1.1 The language of probability distributions

Definition 1

Trial a .... instance of a random experiment e.g. tossing a coin

Experiment .... trials.

Example 1

Give an example of an *experiment*.

Definition 2

Outcome A single, unique, possible .... of a trial experiment

Event A combination of one or more ....

Sample space A set of .... possible outcomes of a trial or experiment

Example 2

What would the original trial/experiment have involved if:

(a) One of the *outcomes* is a 'head'

(b) The *sample space* S is

$$S = \{(H,H,H),(H,H,T),(H,T,H),(T,H,H),\cdots,(T,T,T)\}$$

Definition 3

Population Any set S that we are interested in. The .... of an event or an experiment.

1.2 Random variable

Definition 4

A **random variable** is a .... whose value depends on the outcome of a chance experiment.

Fill in the spaces

When one particular aspect or attribute of a given population S can be measured by a .... (integer, fraction, or any real number), we call that attribute a .... Officially, a random variable X is a function from a population S to numbers.

Definition 5

Discrete random variable can only take on a discrete set of values. Usually associated with .... , and so often take on .... values

Continuous random variable can only take on a continuous range of values. Often associated with .... , and so may take on general rational or floating point decimal values

Example 3

(Pender et al., 2019, p.622) State whether each probability distribution is *numeric* or *categorical*. If it is numeric, state whether it is *discrete* or *continuous*.

1. The number showing when a die is thrown

2. The weight of a randomly chosen adult male in Australia

3. Whether it rains or not on a spring day in Sydney

4. The daily rainfall in Sydney on a September day

5. The colour of a ball drawn from a bag containing four red and three green balls

6. The colours of two balls drawn together from a bag containing four red and three green balls

7. The shoe size of a randomly chosen adult female in Australia

8. ATAR results for a particular year

Answer: **1.** Numeric/discrete **2.** Numeric/continuous **3.** Categorical **4.** Numeric/continuous **5.** Categorical **6.** Categorical **7.** Numeric/can be both discrete and continuous **8.** Numeric/discrete

1.2.1 Relative frequency

Definition 6

If an event E occurs m times out of a total of M times, then the **relative frequency** is ......

Important note

Is *relative frequency* the same as *the probability of*?
See:
http://researchguides.library.vanderbilt.edu/c.php?g=156859&p=1171653

1.3 Probability distributions

Definition 7

The **probability** that the random variable X takes on the value x_i is p_i

....

Definition 8

The collection of values x_i and the probabilities p_i together is referred as a ....

Important note

- The probabilities p_i must always be .... - .... , and
- The sum of the probabilities over the whole population/distribution must be equal to .... .

Example 4

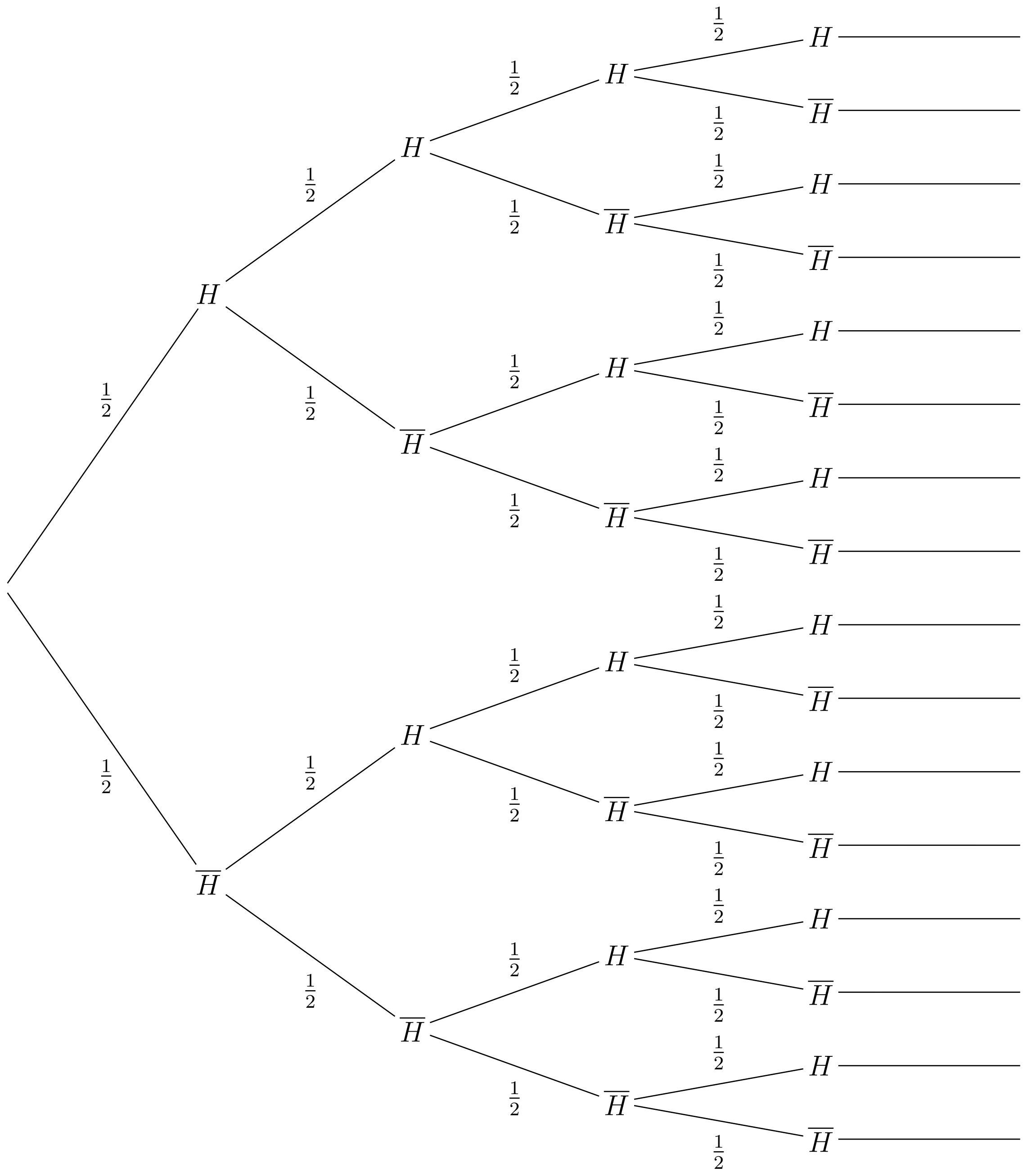
Four coins are thrown and the number of heads is recorded. The number of heads is denoted by X.

(a) Complete the probability distribution table:

x	0	1	2	3	4
$P(X=x)$					

(b) Find:

i. $P(X=3)$ ii. $P(X>2)$ iii. $P(X \text{ is odd})$



Example 5

[2016 VCE Math Methods Paper 2, Q7] The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$P(X=x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

(A) 0.3 (B) 0.305 (C) 0.355 (D) 0.405 (E) 0.8

Answer: (C)

Definition 9

The probability distribution of a random variable can also be described using a **probability distribution function** $P(x)$, also denoted $P(X = x)$.

- The domain of $P(x)$ is the set of possible values of the variable x
- The range of $P(x)$ is the set of values in the probability distribution.

Example 6

(Haese, Haese, & Humphries, 2016, p.188) Show that

$$P(x) = \frac{x^2 + 1}{34} \qquad \text{where } x = 1, 2, 3, 4$$

is a valid probability distribution function.

1.3.1 Uniform probability distribution

Definition 10

A probability distribution is *uniform* if all of its values have the .... i.e. the values are .... possible outcomes.

Example 7

(Pender et al., 2019, p.620) A die is thrown. Write out the probability distribution for the number shown on the die, and graph the distribution.

Further exercises

Ⓐ **Ex 11A**
- All questions

(X1) **Ex 13A**
- All questions

Additional exercises

Source Haese et al. (2016, Ex 7B)

⚠ Some parts rely on content from the Year 12 topic **MA-S2** Descriptive Statistics to be completed first.

1. (a) State whether each of the following is a valid probability distribution

i.

x	1	2	3	4
$P(X = x)$	0.2	0.4	0.15	0.25

ii.

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.2

iii.

x	0	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

iv.

x	2	3	4	5
$P(X = x)$	0.3	0.4	0.5	-0.2

(b) For which of the probability distributions above is X a uniform random variable?

2. Find k in each of these probability distributions:

(a)

x	0	1	2
$P(X = x)$	0.3	k	0.5

(b)

x	0	1	2	3
$P(X = x)$	k	$2k$	$3k$	k

3. Consider the probability distribution:

x	0	1	2	3
$P(X = x)$	0.1	0.25	0.45	a

(a) Find the value of a.

(b) Is X a uniform discrete random variable? Explain your answer.

(c) Find the mode of the distribution.

(d) Find $P(X \geq 2)$.

4. The probability distribution for Jason scoring X home runs in each game during his baseball career is given in the following table:

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

(a) State the value of $P(2)$.

(b) Find the value of a. Explain what this number means.

(c) Find the value of $P(1) + P(2) + P(3) + P(4) + P(5)$. Explain what this means.

(d) Draw a graph of $P(x)$ against x.

(e) ⚠ Find the mode and median of the distribution.

5. A policeman inspected the safety of tyres on cars passing through a checkpoint. The number of tyres X which needed replacing on each car followed the probability distribution below:

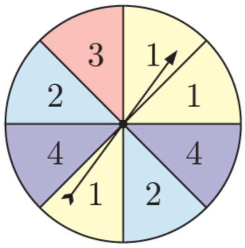
x	0	1	2	3	4
$P(x)$	0.68	0.2	0.06	k	0.02

(a) Find the value of k.

(b) ⚠ Find the mode of the distribution.

(c) Find $P(X > 1)$, and interpret its value.

6. Let X be the result when the spinner alongside is spun.



(a) Display the probability distribution of X in a table.

(b) Graph the probability distribution.

(c) ⚠ Find the mode and median of the distribution.

(d) Find $P(X \leq 3)$

7. 100 people were surveyed about the number of bedrooms in their house. 24 people had one bedroom, 35 had two bedrooms, 27 people had three bedrooms, and 14 people had four bedrooms. Let X be the number of bedrooms a randomly selected person has in their house.

(a) State the possible values of X.

(b) Construct a probability table for X.

(c) ⚠ Find the mode and median of the distribution.

8. A group of 25 basketballers took shots from the free throw line until they scored a goal. 12 of the players only needed one shot, 7 players took two shots, 2 plays took three shots and the rest took four shots. Let X be the number of shots a randomly selected player needs to score a goal.

(a) State the possible values of X.

(b) Construct a probability table for X.

(c) ⚠ Find the mode and median of the distribution.

9. Show that the following are valid probability distribution functions:

(a) $P(x) = \dfrac{x+1}{10}$, for $x = 0, 1, 2, 3$.

(b) $P(x) = \dfrac{6}{11x}$ for $x = 1, 2, 3$

10. Find k for the following probability distribution functions:

(a) $P(x) = k(x+2)$, for $x = 1, 2, 3$.

(b) $P(x) = \dfrac{k}{x+1}$ for $x = 0, 1, 2, 3$

11. A discrete random variable X has the probability distribution function $P(x) = \dfrac{4x - x^2}{a}$ for $x = 0, 1, 2, 3$.

(a) Find the value of a.

(b) Find $P(X = 1)$.

(c) ⚠ Find the mode of the distribution.

12. Two tickets are randomly selected without replacement from a bag containing 5 blue and 3 green tickets. Let X denote the number of blue tickets selected.

Find the probability distribution of X.

13. A hat contains 2 red balls and 2 green balls. Balls are randomly selected without replacement until a green ball is selected. Let X denote the total number of balls selected.

(a) State the possible values of X.

(b) Find the probability distribution of X.

14. When a pair of dice is rolled, S denotes the sum of the top faces.

(a) Display the possible results in a table.

(b) Find the probability distribution of S.

(c) Find the mode of the distribution.

(d) Find $P(S \geq 10)$.

15. (a) The exponential function e^x can be defined as a power series as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Use this definition to evaluate

$$\sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!}$$

(b) Suppose X is the number of cars that pass a shop between 3:00 pm and 3:03 pm. The probability distribution for X is given by

$$P(X = x) = \frac{(0.2)^x e^{-0.2}}{x!}$$

where $x = 0, 1, 2, 3 \cdots$.

i. Explain why this is a valid probability distribution.

ii. Evaluate $P(X = 0)$, $P(X = 1)$ and $P(X = 2)$.

iii. Find the probability that at least three cars will pass the shop in a given period.

Answers

1. (a) i. Yes ii. No iii. Yes iv. No

(b) (a) iii, X is a uniform random variable.

2. (a) $k = 0.2$ (b) $k = \frac{1}{7}$

3. (a) $a = 0.2$

(b) No, as the probabilities of each outcome are not all equal.

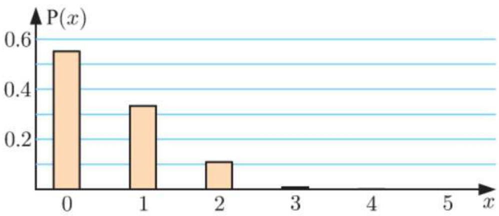
(c) 2

(d) $P(X \geq 2) = 0.65$.

4. (a) $P(2) = 0.1088$

(b) $a = 0.5488$ is the probability that Jason does not hit a home run in a game.

(c) $P(1) + P(2) + P(3) + P(4) + P(5) = 0.4512$ and is the probability that Jason will hit one or more home runs in a game.

(d) 

(e) mode = 0 home runs, median = 0 home runs..

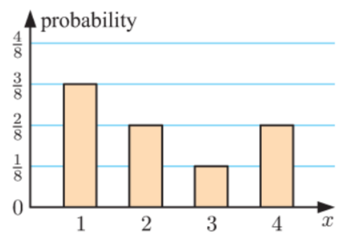
5. (a) $k = 0.04$

(b) 0 tyres

(c) $P(X > 1) = 0.12$ which is the probability that 2, 3 or 4 tyres will need replacing on a car being inspected

6. (a)

x	1	2	3	4
$P(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

(b) 

(c) mode = 1, median = 2

(d) $P(x \leq 3) = \frac{3}{4}$

7. (a) $X = 1, 2, 3$ or 4

(b)

x	1	2	3	4
$P(X = x)$	0.24	0.35	0.27	0.14

(c) mode = 2 bedrooms, median = 2 bedrooms.

8. (a) $X = 1, 2, 3$ or 4

(b)

x	1	2	3	4
$P(X = x)$	0.48	0.28	0.08	0.16

(c) mode = 1 shot, median = 2 shots.

9. (a) $P(0) = \frac{1}{10}$, $P(1) = \frac{2}{10}$, $P(2) = \frac{3}{10}$, $P(3) = \frac{4}{10}$, $0 \leq P(x_i) \leq 1$ in each case, and

$$\sum P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$$

Hence $P(x)$ is a valid probability distribution function.

(b) $P(1) = \frac{6}{11}$, $P(2) = \frac{6}{22}$, $P(3) = \frac{6}{33}$. $0 \leq P(x_1) \leq 1$ in each case, and

$$\sum P(X_i) = \frac{6}{11} + \frac{6}{22} + \frac{6}{33} = 1$$

Hence $P(x)$ is a valid probability distribution function.

10. (a) $k = \frac{1}{12}$ (b) $k = \frac{12}{25}$

11. (a) $a = 10$ (b) $P(X = 1)$(c)= 2 $\frac{3}{10}$

12.

x	0	1	2
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

13. (a) $X = 1$, 2 or 3

(b)

x	1	2	3
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

14. (a)

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(b)

x	2	3	4	5	6	7	8	9	10	11	12
$P(S = s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(c) 7 (d) $P(S \geq 10) = \frac{1}{6}$

15. (a) $\sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!} = 1$

(b) i. $\sum P(x_i) = 1$

ii.
- $P(X = 0) = e^{-0.2} \approx 0.819$
- $P(X = 1) = 0.2e^{-0.2} \approx 0.164$
- $P(X = 2) = 0.02e^{-0.2} \approx 0.0164$

iii. $P(X \geq 3) = 0.00115$

Section 2

Expected value

Fill in the spaces

Given a probability distribution for a random variable X,

Mean (Expected value) describes where the distribution is ....

Variance describes how .... from the mean the distribution is.

Standard deviation is a secondary measure, and it is just the .... of the variance.

2.1 Expected value

Definition 11

The *expected value*, ..., is the mean of the theoretical distribution and is the sum of the product of each outcome x_i, with its associated probability, p_i.

Laws/Results

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i$$

Example 8

Consider the experiment Example 4 on page 8, *toss four coins and count the number of heads.*

x	0	1	2	3	4
$P(X = x)$					

Find the expected value $E(X)$ **Answer:** 2

Example 9

(Pender et al., 2019, Example 3, p.631) Twenty friends go out to dinner at a Vietnamese restaurant. One has \$560 cash, four have \$350 cash, two have \$180 cash, three have \$80 cash, four have \$50 cash, and one has \$40 cash. Five have no cash, and are expecting to borrow from the others. An armed robber bursts in and seizes one of the friends at random. She threatens him, grabs all his cash, and runs away. Construct a probability distribution table, and find her expected criminal gain. **Answer:** \$140

Example 10

(Pender et al., 2019, Example 4, p.632) A regular six sided die is thrown.

(a) Graph the distribution

(b) Find its expected value.

Answer: 3.5

Example 11

[2018 VCE Mathematical Methods Paper 2, Q12] The discrete random variable X has the following probability distribution.

x	0	1	2	3	6
$P(X = x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X. $P(X < \mu)$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{17}{20}$ (D) $\frac{4}{5}$ (E) $\frac{7}{10}$

Answer: E

Example 12

[2016 VCE Mathematical Methods Paper 2, Q19] Consider the discrete probability distribution with random variable X shown in the table below.

x	-1	0	b	$2b$	4
$P(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $E(X)$ are respectively

(A) -0.8 and 1

(B) -0.8 and 1.6

(C) 0 and 2.4

(D) 0.2125 and 1

(E) 0 and 1

Answer: E

Example 13

[2015 VCE Mathematical Methods Paper 2, Q14] Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$P(X = x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

(A) 2 (B) 3 (C) $\frac{7}{2}$ (D) $\frac{11}{3}$ (E) 4

Answer: D

Example 14

[2012 VCE Mathematical Methods Paper 1, Q4] On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$P(X = x)$	0.2	0.2	0.5	0.1

(a) Find the mean of X

(b) What is the probability that Daniel receives only one telephone call on each of three consecutive days?

(c) Daniel receives telephone calls on both Monday and Tuesday. What is the probability that Daniel receives a total of four calls over these two days?

Answer: (a) 1.5 (b) 0.008 (c) $\frac{29}{64}$

Example 15

[2013 VCE Mathematical Methods Paper 1, Q17] The probability distribution of a discrete random variable, X, is given by the table below

x	0	1	2	3	4
$P(X=x)$	0.2	$0.6p^2$	0.1	$1-p$	0.1

(a) Show that $p = \frac{2}{3}$ or $p = 1$

(b) Let $p = \frac{2}{3}$

i. Calculate $E(X)$

ii. Find $P(X \geq E(X))$

Answer: (a) Show (b) $\frac{28}{15}$ (c) $\frac{8}{15}$

2.1.1 Additional exercises

Source Haese et al. (2016, Ex 7C.2 & 7C.3)

1. Find $E(X)$ for the following probability distributions:

(a)

x_i	1	2	3
p_i	0.4	0.5	0.1

(b)

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

(c)

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

(d)

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

2. Consider the following probability distribution:

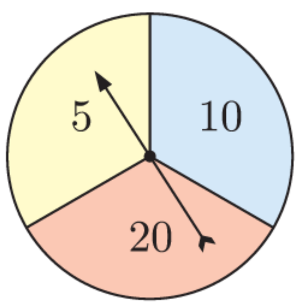
x_i	1	3	5
$P(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

(a) Find the value of a.

(b) Find the mode of the distribution.

(c) Find the mean μ of the distribution.

3. When the spinner alongside is spun, players are awarded the resulting number of points. In the longer term, how many points can we expect to be awarded per spin?



4. When Ernie goes fishing, he catches 0, 1, 2, or 3 fish, with the probabilities shown. On average, how many fish would you expect Ernie to catch per fishing trip?

Number of fish	0	1	2	3
Probability	0.17	0.28	0.36	0.19

5. Each time Pam visits the library, she borrows either 1, 2, 3, 4, or 5 books, with the probabilities as shown.

Number of books	1	2	3	4	5
Probability	0.16	0.15	a	0.28	0.16

(a) Find the value of a.

(b) Find the mode of the distribution.

(c) On average, how many books does Pam borrow per vlSlt.

6. Lachlan randomly selects a ball from a bag containing 5 red balls 2 green balls, and 1 white ball. He is then allowed to take a number of lollies from a lolly jar. The number of lollies is determined by the colour of ball, as shown in the table.

Colour	Number of lollies
Red	4
Green	6
White	10

Find the average number of lollies that Lachlan would expect to receive.

7. When ten-pin bowler Jenna bowls her first bowl of a frame, she always knocks down 8 pins.
$\frac{1}{3}$ of the time she knocks down 8 pins, and $\frac{2}{5}$ of the time she knocks down 9 pins.

(a) Find the probability that she knocks down all 10 pins on the first bowl.

(b) On average, how many pins does Jenna expect to knock down with her first bowl?

8. Given $E(X) = 2.5$, find a and b:

x	1	2	3	4
$P(X = x)$	0.3	a	b	0.2

9. When Brad's soccer team plays an offensive strategy, they win 30% of the time and lose 55% of the time.

When they play a defensive strategy, they win 20% of the time and lose 30% of the time.

On the league table, teams are awarded 3 points for a win, 1 point for a draw, and no points for a loss.

(a) Find the probability that Brad's team will draw a match under each strategy.

(b) Calculate the expected number of points per game under each strategy.

(c) In the long run, is it better for the team to play an offensive, or defensive strategy?

(d) Should the strategy change if teams are awarded 4 points instead of 3 points for a win?

10. Every Thursday, Zoe meets her friends in the city for dinner. There are two car parks nearby, the costs for which are shown below:

Car Park A

Time	Cost
0 - 1 hr	\$7
1 - 2 hrs	\$12
2 - 3 hrs	\$15
3 - 4 hrs	\$19

Car Park B

Time	Cost
0 - 1 hr	\$6.50
1 - 2 hrs	\$11
2 - 3 hrs	\$16
3 - 4 hrs	\$18.5

Zoe's dinner takes 1 - 2 hrs 20% of the time, 2 - 3 hrs 70% of the time, and 3 - 4 hrs 10% of the time.

(a) Which car park is cheapest for Zoe if she stays:

i. 1-2 hrs

ii. 2-3 hrs

iii. 3-4 hrs

(b) When Zoe parks her car, she does not know how long she will stay. Which car park do you recommend for her? Explain your answer.

11. An insurance policy covers a \$20 000 sapphire ring against theft and loss. If the ring is stolen then the insurance company will pay the policy owner in full. If the ring is lost then they will pay the owner \$8000. From past experience, the insurance company knows that the probability of theft is 0.002 5, and the probability of loss is 0.03. How much should the company charge to cover the ring in order that their expected return is \$100?

Questions from here onwards asks whether a game is *fair* or not. If X represents the gain of the player from each game, then the game is *fair* if $E(X) = 0$ (i.e. no win or loss)

12. A dice game cost \$2 to play If an odd number is rolled, the player receives \$3. If an even number is rolled, the player receives \$1.

Determine whether the game is fair.

13. A man rolls a regular six-sided die. He wins the number of dollars shown on the uppermost face.

(a) Find the expected return from one roll of the die.

(b) Find the expected gain if it costs \$4 to play the game.

(c) Would you advise the man to play many games?

14. A roulette wheel has 18 red numbers, 18 black numbers, and 1 green number. Each number has an equal chance of occurring. I place a bet of \$2 on red. If a red is spun, I receive my \$2 back plus another \$2. Otherwise I lose my \$2.

(a) Calculate the expected gain from this bet.

(b) If the same bet is made 100 times, what is the expected result?

15. A person pays \$5 to play a game with a pair of coins . If two heads appear then \$10 is won. If a head and a tail appear then \$3 is won. If two tails appear then \$1 is won. Let X be the gain of the person from each game. Find the expected value of X.

16. In a carnival game, a player randomly selects a ticket from a box of tickets numbered 1 to 20. If the selected number is a multiple of 3, the player wins 5 tokens. If the selected number is a multiple of 10 the player wins 10 tokens.

(a) Calculate the probability of a player winning:

i. 5 tokens

ii. 10 tokens

(b) Let X be the number of tokens won from playing this game. Find the expected value of X.

(c) If it costs 3 tokens to play the game, would you recommend playing the game many times? Explain your answer.

17. A person selects a disc from a bag containing 10 black discs, 4 blue discs, and 1 gold disc. They win \$1 for a black disc, \$5 for a blue disc, and \$20 for the gold disc. The game costs \$4 to play.

(a) Calculate the expected gain for this game , and hence show that the game is not fair.

(b) To make the game fair, the prize money for selecting the gold disc is increased. Find the new prize money for selecting the gold disc.

18. At a charity event there is a money-raising game involving a pair of ordinary dice . The game costs $\$a$ to play. When the two dice are ro11ed, their sum is described by the variable X. A sum which is less than 4 or between 7 and 9 inclusive gives a return of $\$\frac{a}{3}$. A result between 4 and 6 inclusive gives a return of \$7 . A result of 10 or more gives a return of \$21.

(a) Determine

i. $P(X \leq 3)$

ii. $P(4 \leq X \leq 6)$

iii. $P(7 \leq X \leq 9)$

iv. $P(X \geq 10)$.

(b) Show that the expected gain of a player is given by $\dfrac{1}{6}(35 - 5a)$ dollars.

(c) What value would a need to have for the game to be 'fair'?

(d) Explain why the organisers would not let a be 4.

(e) The organisers set $a = 9$ for the event, and the game is played 2 406 times . Estimate the amount of money raised by this game.

19. In a fundraising game "Lucky 11", a player selects 3 cards without replacement from a box containing 5 red, 4 blue, and 3 green cards.

The player wins \$11 if the cards drawn are all the same colour or are one of each colour.

If the organiser of the game wants to make an average of \$1 per game, how much should they charge to play it?

Answers on page 34.

2.2 Linearity property of $E(X)$

Laws/Results

Linearity If X is a random variable,

$$E(aX + b) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{where } a, b \in \mathbb{R}$$

Steps

Proof - ⚠ Not in the syllabus, but provided as an interesting prelude to the alternative formula for the variance on page 29

1. $E(X) = \sum \ldots\ldots\ldots$

2. $E(aX + b) = \sum \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

3. Expand p_i into $\ldots\ldots\ldots\ldots\ldots\ldots$:

4. Note $\sum p_i = \ldots$:

Hence

$$E(aX + b) = aE(x) + b$$

Further exercises

Ⓐ **Ex 11B**

- All questions

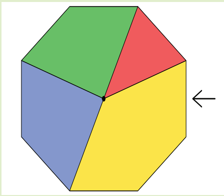
(X1) **Ex 13B**

- All questions

Example 16

[UNSW Teacher's PD Day, May 2019] Extended example to consolidate all of the work so far:

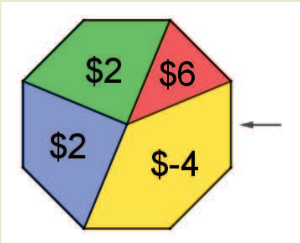
(a) For the spinner represented below, what is the sample space ? What is the probability associated to each outcome of the sample space?



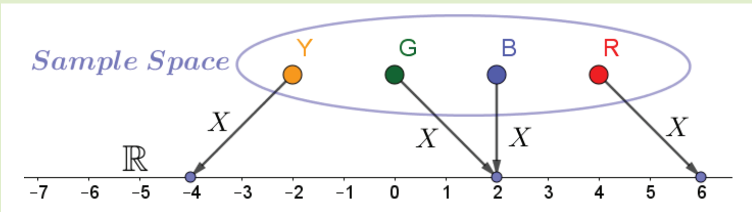
What type of data forms the sample space?

(b) With the same spinner, the rules of the game are now defined:

- Playing consists of choosing one of the colours.
- The player wins \$6 if red is spun.
- \$2 is won if green or blue are spun.
- \$4 is lost if yellow is spun.



Draw a probability distribution table with the random variable X being the net financial gain if the spinner is spun once.



(c) Calculate $E(X)$, and describe in words what $E(X)$ means in this situation.

Answer: 0.25

(d) The same game is played in Japan and the amounts that players win or lose are the same as in the Australian version of the game, except for the fact that they are now expressed in Japanese yen (¥). Another difference is that now the players need to pay ¥60 each time they play.

Let Y be the net amount won when a player plays once in Japan, expressed in Japanese yen.

i. If \$1 converts to ¥80, express Y in terms of X.

ii. Find $E(Y)$ given the value of $E(X)$ above. **Answer:** -40

iii. Are the Australian or Japanese versions of this game *fair*?

Definition 12

A *fair* game has an expected value of .... for its financial return.

Section 3

Measures of spread

3.1 Variance

Fill in the spaces

Variance and *standard deviation*, which are two measures of spread, will be explored in this section.

- Variance is easier to define and calculate
- Standard deviation (the .... of the variance) is important as it has the same units as the values of the ....

Definition 13

The *variance* is the mean (....) of the squared-distance of the scores from the mean. It has symbol ...... and found by

$$\begin{aligned}\operatorname{Var}(X) = \sigma^2 &= \frac{\sum (x_i - \overline{x})^2 f_i}{n} \\ &= \sum (x-\mu)^2 p(x) \\ &= E((X-\mu)^2)\end{aligned}$$

Important note

To find the variance, take the difference of each value from the mean, i.e. $x - \mu$. The difference is called the .... of the value x from the mean μ. Then we square the deviation to give $(x-\mu)^2$.

Example 17

Return again to the number of heads when four coins are tossed (Example 4 on page 8)

x	0	1	2	3	4
$P(X = x)$					

In the last section, the expected value of the distribution was $\mu = 2$. What is the variance $\text{Var}(X)$?

Hint: Construct a table with $x, p(x), (x-\mu)^2$ and $(x-\mu)^2 p(x)$

Answer: 1

Example 18

(Pender et al., 2019, Example 5, p.638) A die is thrown, producing a uniform distribution. Find the mean and variance of the distribution.

Answer: mean: $3\frac{1}{2}$, variance: $\frac{35}{12}$

Laws/Results

The alternative formula of the variance

$$\text{Var}(X) = E(X^2) - \mu^2$$

Steps

Proof for $\text{Var}(X) = E((X-\mu)^2) = E(X^2) - \mu^2$

$$\text{Var}(X) = E\left((X-\mu)^2\right)$$

Example 19

(Identical to Example 18 on the facing page)
A die is thrown, producing a uniform distribution. Find the variance of the distribution using the alternative formula $\text{Var}(X) = E(X^2) - \mu^2$.

Hint: Construct a table with $x, p(x), xp(x)$ and $x^2p(x)$

Example 20

[2017 VCE Math Methods Paper 2, Q14] The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$P(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

(A) $2p(1 - 3p)$

(B) $1 - 4p$

(C) $(1 - 3p)^2$

(D) $6p - 16p^2$

(E) $p(5 - 9p)$

Answer: D

3.2 Standard deviation

Definition 14

The **standard deviation** is the .... of the variance.
Symbol: ...

Important note

1. The standard deviation has the same .... as the values.
2. The variance and the standard deviation are measures of ....

Example 21

(Pender et al., 2019, Question 3, p.642). For each random variable, calculate $E(X)$, $\text{Var}(X)$ twice using the two formulae, and standard deviation.

(a)

x	0	1	2	3	4
$p(x)$	0.0	0.1	0.2	0.3	0.4

(b)

x	-2	-1	0	1	2
$p(x)$	0.3	0.1	0.2	0.1	0.3

Answer: (a) 3, 1, 1 (b) 0, 2.6, 1.6

Example 22

(Pender et al., 2019, Question 11, p.645). For some values of k, a random variable X with values $1, 2, 3, 4$ is defined by

$$P(X = x) = kx$$

for $x = 1, 2, 3, 4$.
Find k, then find the expected value and the standard deviation.

Answer: $k = \frac{1}{10}$, $E(X) = 3$, $\sigma = 1$

Further exercises

Ⓐ **Ex 11C**

- All questions

Ⓐ **Chapter 11 Review exercise**

- All questions

(X1) **Ex 13C**

- All questions

(X1) **Chapter 13 Review Exercise**

- All questions

3.2.1 Additional exercises

Source Haese et al. (2016, Ex 7D)

1. For each probability distribution, find the:

i. mean μ

ii. variance σ^2

iii. standard deviation σ

(a)

x	1	2	3
$P(X = x)$	0.3	0.4	0.3

(b)

x	0	1	2	3
$P(X = x)$	0.2	0.4	0.1	0.3

2. Consider the probability distribution:

x_i	2	4	10	20
p_i	k	0.05	0.35	$3k$

(a) Find the value of k.

(b) Find the mode of the distribution.

(c) Find the mean μ.

(d) Find the standard deviation σ.

3. The probability distributions below refer to the number of aces served by Michelle and Amanda in each set of tennis they play. Let M be the random variable for the number of aces scored by Michelle and A be the random variable for the number of aces scored by Amanda.

- Michelle:

a	0	1	2	3	4
$P(A = a)$	0.1	0.15	0.45	0.25	0.05

- Amanda:

m	0	1	2	3	4
$P(M = m)$	0.2	0.1	0.35	0.2	0.15

Find the standard deviation of each data set. Which player has the greater variation in the number of aces served?

4. A country exports crayfish to overseas markets. The buyers are prepared to pay high prices when the crayfish arrive still alive.

Let X be the number of deaths per dozen crayfish. The probability distribution for X is given by:

x_i	0	1	2	3	4	5	> 5
p_i	0.54	0.26	0.15	k	0.01	0.01	0.00

(a) Find k.

(b) Over a long period, what is the mean number of deaths per dozen crayfish?

(c) Find the standard deviation for the distribution.

5. A die is numbered 1, 1, 2, 3, 3, 3. Let X be the result when the die is rolled once.

(a) Construct the probability distribution for X.

(b) Find the mean μ.

(c) Find the standard deviation for the distribution.

6. A random variable X has the probability distribution function $P(x) = \dfrac{x^2 + x}{20}$, $x = 1, 2, 3$. For this distribution, calculate the:

(a) mean μ

(b) standard deviation σ

7. Suppose X is a uniform discrete random variable with possible values $X = 1, 2, 3, \cdots, n$. Show that the variance of X is $\dfrac{n^2 - 1}{12}$.

Hint:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Answers to Additional Exercises, Section 2.1.1 on page 21

1. (a) 1.7 (b) 2.5 (c) 3.85 (d) 30 **2.** (a) $a = \frac{1}{2}$ (b) 3 (c) $\mu = \frac{12}{5}$ **3.** ≈ 11.7 points **4.** 1.57 fish **5.** (a) $a = 0.25$ (b) 4 books (c) 3.13 books **6.** 5.25 lollies **7.** (a) $\frac{4}{15}$ (b) ≈ 8.93 pins **8.** $a = 0.1$, $b = 0.4$ **9.** (a) offensive: $P(\text{draw}) = 0.15$, defensive: $P(\text{draw}) = 0.5$ (b) offensive: 1.05 points per game, defensive: 1.1 points per game (c) defensive (d) Yes, offensive would be better. **10.** (a) i. B ii. A iii. B (b) Zoe should choose car park A as the expected cost for car park A is \$14.8, and expected cost for car park B is \$15.25. **11.** \$390 **12.** fair **13.** (a) \$3.50 (b) −\$0.50 (c) no **14.** (a) ≈ −\$0.05 (b) lose ≈ \$5.41 **15.** −\$0.75 **16.** (a) i. 0.3 ii. 0.1 (b) $E(X) = 2.5$ (c) No, as the player can expect to lose half a token on average per game. **17.** (a) Expected gain ≈ −\$0.67 ≠ \$0 (b) \$30 **18.** (a) i. $P(X \le 3) = \frac{1}{12}$ ii. $P(4 \le X \le 6) = \frac{1}{3}$ iii. $P(7 \le X \le 9) = \frac{5}{12}$ iv. $P(X \ge 10) = \frac{1}{6}$ (b) Show (c) $a = 7$ (d) The organisers would lose \$2.50 per game (e) \$4 010 **19.** \$4.75

Answers to Additional Exercises, Section 3.2.1 on the preceding page

1. (a) i. $\mu = 2$ ii. $\sigma^2 = 0.6$ iii. $\sigma \approx 0.775$ (b) i. $\mu = 1.5$ ii. $\sigma^2 = 1.25$ iii. $\sigma \approx 1.12$ **2.** (a) $k = 0.15$ (b) 20 (c) $\mu = 13$ (d) $\sigma \approx 6.88$ **3.** $\sigma_M = 1$ ace, $\sigma_A \approx 1.30$ aces. Amanda has the greater variation. **4.** (a) $k = 0.03$ (b) $\mu = 0.74$ (c) $\sigma \approx 0.996$ **5.** (a)

x	1	2	3
$P(X = x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

(b) $\mu \approx 2.17$ (c) $\sigma \approx 0.898$ **6.** (a) $\mu = 2.5$ (b) $\sigma \approx 0.671$

Section 4

Sampling

4.1 Sample vs theoretical/population

Fill in the spaces

- When collecting data, the resulting .... yields a .... (frequency table).
- Relative frequencies f_r in the .... corresponds to the probabilities $P(X = x)$ in a ....

Definition 15

Sample mean

$$\overline{x} = \ldots\ldots\ldots\ldots\ldots$$

where f_r is the relative frequency

Important note

- Analogous to the .... in theoretical probability
- $\overline{x}$ will often not be equal to μ.

Definition 16

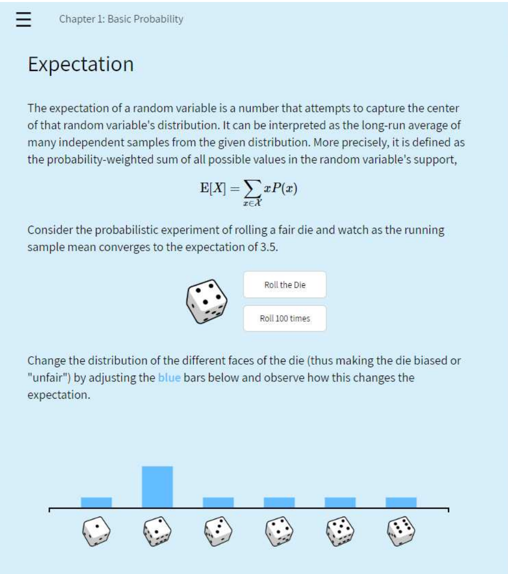
Sample variance

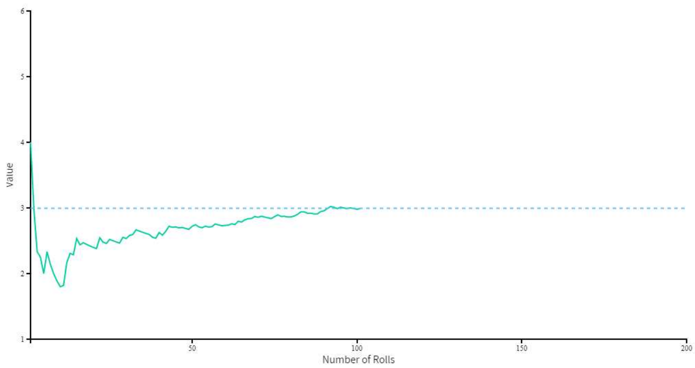
$$s^2 = \text{..............................} = \text{..............................}$$

Important note

- Analogous to the variance in theoretical probability
- s^2 will often not be equal to σ^2.

Navigate to Seeing Theory and read up on Chapter 1. Play with the sliders and options (to make biased dice!) there to see the difference between the relative frequency and theoretical probability.





Further exercises

(A) Ex 11D, (x1) Ex 13D

- Investigate how to use R to simplify the generation of data. Download for your computer here: `https://www.r-project.org/`
- Exercises here are not the usual type - most are investigative.

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$

$S = \frac{a}{1 - r}, \; |r| < 1$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

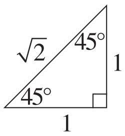
$\log_a x = \frac{\log_b x}{\log_b a}$

$a^x = e^{x \ln a}$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

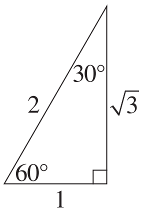
$A = \dfrac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

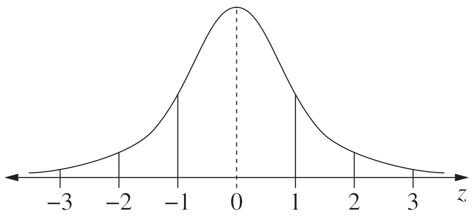
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

References

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